Parametric amplification of spatiotemporal localized envelope waves

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The process of parametric amplification of dispersionless and diffractionless spatiotemporal envelope waves in nonlinear $\chi^{(2)}$ media is theoretically investigated in the undepleted pump limit and assuming a continuous-wave (cw) plane wave pump. By deriving an extended paraxial envelope equation for the signal field, we show that distortionless and efficient parametric amplification is possible for different types of localized waves, including pulsed Bessel beams, envelope X waves and sinc-shaped envelope waves. An analytic expression of the spectral parametric gain for polychromatic Bessel beams is also derived beyond the paraxial approximation and takes into account higher-order dispersion effects.

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I. INTRODUCTION

The study and the generation of nondiffracting spatial-temporal localized electromagnetic or acoustic waves, such as X waves, focus wave modes, pulsed Bessel beams, etc., have attracted considerable and increasing interest in recent years (see, e.g., [1–10]), and their potential impact for applications in optical communications, metrology, spectroscopy, and imaging has been pointed out. In the optical context, recent investigations have been focused on the existence of localized *envelope* light waves propagating without spreading both in space and time in linear dispersive media as a result of spatial-temporal coupling effects [11–18]. These localized envelope waves are usually constructed as a superposition of monochromatic Bessel beams with a frequency-dependent cone angle to allow for the simultaneous cancellation of both diffraction and dispersion at any order, leading to undistorted propagation of the envelope light wave at a group velocity which in general is distinct from the phase velocity of the carrier [11]. By specializing the general form of polychromatic Bessel beams originally proposed in Ref. [11], the existence of different families of nondiffracting and nondispersive envelope waves propagating in dispersive transparent and linear media has been recently pointed out, including pulsed Bessel beams [13,15], luminal envelope X waves [16], Gauss-Laguerre waves [17], and subluminal sinc-shaped envelope waves [18].

Though space-time wave localization is a purely linear phenomenon and polychromatic Bessel beams can be generated in the purely linear optics realm (by means of, e.g., axicon and diffraction gratings or computer holograms), their relevance in nonlinear optical processes has been remarkably studied in a series of recent works [19–23], which have opened the interesting field of nonlinear optics of spatial-temporal localized waves. Spontaneous generation of polychromatic Bessel beams and X-type waves mediated by a conical emission process seems to be in fact a rather general phenomenon in nonlinear optical processes. In particular, by considering the processes of second-harmonic generation in $\chi^{(2)}$ media and the spatial-temporal conical instability in $\chi^{(3)}$ Kerr media, it has been shown that the generalized phase-matching conditions underlying the nonlinear interaction process spontaneously produce the proper dispersion curve of the Bessel cone angle supporting envelope X-type waves. The aim of this paper is to study the process of parametric amplification of localized envelope waves, at the fundamental carrier frequency ω_0, in nonlinear dispersive $\chi^{(2)}$ media pumped by a cw plane-wave pump field at frequency $2\omega_0$. By deriving an extended paraxial wave equation for the signal field, it is shown that efficient and distortionless amplification of different types of paraxial localized envelope waves is possible due to the simultaneous achievement of angular and spectral phase matching requirements. The general form of the spectral parametric gain for polychromatic Bessel beams, beyond the quasimonochromatic and paraxial approximations, is also derived and the limits for distortionless amplification are discussed. The paper is organized as follows. In Sec. II the basic wave equations describing parametric amplification in a nonlinear $\chi^{(2)}$ dispersive medium are reviewed, and an extended envelope paraxial wave equation, which includes parametric gain, material dispersion up to second-order, and diffraction in the paraxial limit, is derived in the undepleted pump limit. In Sec. III the process of parametric amplification is analytically studied, and distortionless parametric amplification of three different types of localized envelope waves, namely pulsed Bessel beams, envelope X waves and sinc-shaped waves, is proven. The general expression of the spectral parametric gain is also derived for polychromatic Bessel beams beyond the paraxial limit and takes into account dispersion effects at any order. Some numerical results are then presented, and the connection between the parametric instability and envelope localized waves is briefly addressed. Finally, in Sec. IV the main conclusions are outlined.

II. PROPAGATION OF SPATIOTEMPORAL WAVES IN DISPERSIVE MEDIA WITH PARAMETRIC GAIN: GENERAL

A. Description of the model and the parametric wave equation

The starting point of the analysis is the propagation scalar wave equation for a linearly polarized electric field $\mathcal{E}(x,y,z,t)$ in a nonlinear $\chi^{(2)}$ medium taking into account

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the material dispersion at any order and without making any paraxial approximation. Such an equation reads (see, for instance, [24]):

$$\frac{\partial^2 \mathcal{E}}{\partial z^2}+\nabla_\perp^2 \mathcal{E}+\int_{-\infty}^{\infty} d\omega k^2(\omega)\hat{\mathcal{E}}(\omega)\exp(-i\omega t)=\mu_0\frac{\partial^2 \mathcal{P}^{NL}}{\partial t^2}, \tag{1}$$

where z is the propagation direction of the localized wave; $\nabla_\perp^2=\partial^2/\partial x^2+\partial^2/\partial y^2$ is the transverse Laplacian; $k(\omega)=(\omega/c_0)n(\omega)$ is the dispersion relation defined by the linear refractive index $n(\omega)$; c_0 is the speed of light in vacuum; μ_0 is the vacuum magnetic permeability; $\hat{\mathcal{E}}(\omega)=(2\pi)^{-1}\int d\omega \mathcal{E}(t)\exp(i\omega t)$ is the temporal Fourier transform of $\mathcal{E}(t)$; and $\mathcal{P}^{NL}$ is the nonlinear driving polarization term. For a quadratic medium and neglecting dispersion effects of second-order polarization, one can assume $\mathcal{P}^{NL}=\epsilon_0\chi^{(2)}(z)\mathcal{E}^2$, where $\chi^{(2)}(z)$ is the relevant nonlinear susceptibility term involved in the nonlinear interaction, which in general is assumed to vary along the propagation direction z to account for a quasi-phase-matching (QPM) grating structure. To study parametric amplification, we assume that the field $\mathcal{E}(t)$ can be described by the superposition of a weak quasimonochromatic signal wave at carrier frequency ω_0 (the envelope localized wave to be amplified) and a strong pump field at the carrier frequency $2\omega_0$, i.e., we assume

$$\mathcal{E}(x,y,z,t)=\frac{1}{2}[\mathcal{E}_1(x,y,z,t)\exp(-i\omega_0 t)+\mathcal{E}_2(x,y,z,t)\exp(-2i\omega_0 t)+\text{c.c.}], \tag{2}$$

where the amplitudes $\mathcal{E}_{1,2}$ of the two waves are assumed to vary slowly with respect to time t as compared to the exponential terms. Substitution of Eq. (2) into Eq. (1) and setting equal the terms oscillating at the same frequency yields the following coupled-wave equations:

$$\frac{\partial^2 \mathcal{E}_1}{\partial z^2}+\nabla_\perp^2 \mathcal{E}_1+k^2\left(\omega_0+i\frac{\partial}{\partial t}\right)\mathcal{E}_1=-\chi^{(2)}\left(\frac{\omega_0}{c_0}\right)^2\mathcal{E}_1^*\mathcal{E}_2 \tag{3}$$

$$\frac{\partial^2 \mathcal{E}_2}{\partial z^2}+\nabla_\perp^2 \mathcal{E}_2+k^2\left(2\omega_0+i\frac{\partial}{\partial t}\right)\mathcal{E}_2=-2\chi^{(2)}\left(\frac{\omega_0}{c_0}\right)^2\mathcal{E}_1^2. \tag{4}$$

In deriving the previous equations, we neglected the nonresonant terms in the nonlinear polarization driving term $\mathcal{P}^{NL}$ and used the following identity:

$$\int_{-\infty}^{\infty} d\omega k^2(\omega)\hat{\mathcal{E}}(\omega)\exp(-i\omega t)=\left[k^2\left(\omega_0+i\frac{\partial}{\partial t}\right)\mathcal{A}(t)\right]\exp(-i\omega_0 t), \tag{5}$$

which is valid for any signal of the form $\mathcal{E}(t)=\mathcal{A}(t)\exp(-i\omega_0 t)$, where the operator $k^2(\omega_0+i\partial_t)$ is defined through the power series expansion of $k^2(\omega)$ around $\omega=\omega_0$ after the substitution $(\omega-\omega_0)\to i\partial_t$. In the following, we will study the propagation of a weak signal field at carrier frequency ω_0 in the presence of a strong plane-wave and continuous-wave pump field of amplitude E_2 in a forward interaction scheme, and assume that the material is transparent in the spectral region of operation for both the pump and the signal fields. In the undepleted pump approximation, we may hence assume $\mathcal{E}_2=E_2\exp(ik_2 z)$, where $k_2=k(2\omega_0)$, and the propagation equation for the signal wave [Eq. (3)] reads

$$\frac{\partial^2 \mathcal{E}_1}{\partial z^2}+\nabla_\perp^2 \mathcal{E}_1+k^2\left(\omega_0+i\frac{\partial}{\partial t}\right)\mathcal{E}_1=-\sigma(z)\mathcal{E}_1^*\exp(ik_2 z), \tag{6}$$

where we have set $\sigma(z)\equiv\chi^{(2)}(z)(\omega_0/c_0)^2E_2$. Equation (6) is the basic nonparaxial wave equation that governs dispersive wave propagation with a parametric gain term.

B. The extended paraxial envelope equation

In the case of propagation of quasimonochromatic and paraxial beams, which is a rather typical experimental condition, the wave equation (6) can be simplified by accounting for material dispersion up to second-order and diffraction effects in the paraxial approximation. When these assumptions are made for the initial coupled-wave equations (3) and (4), one ends up with standard coupled mode equations of field envelopes for the fundamental and second-harmonic fields (see, for instance, [19]), or just with the envelope equation for the signal field in the undepleted pump limit. Formally, the envelope equation for the signal field at frequency ω_0 can be obtained by a multiple scale analysis of Eq. (6) (see, e.g., [24]). In the standard derivation, one assumes that at leading order the carrier wave number is given by $k_1=k(\omega_0)$, corresponding to a phase velocity $v_f=\omega_0/k_1$, and the envelope propagates with a group velocity $v_g=1/k_1'$, where $k_1'=(\partial k/\partial\omega)_{\omega_0}$. However, even in the paraxial approximation, it is known that the phase and group velocities of localized envelope waves propagating in dispersive media may differ, though by a small amount, from the previous values to allow for simultaneous cancellation of dispersion and diffraction (see, e.g., [11,13,14,18]). In order to account for slight group and phase velocity shift effects, one can derive a generalized envelope equation by an extension of the multiple-scale method by fixing *a priori*, at leading order in the analysis, the carrier wave number k_z and group velocity $v_g=1/k_z'$ of the envelope, where k_z and k_z' are close to, but not necessarily coincident with, k_1 and k_1', respectively. After setting $\mathcal{E}_1=E_1(x,y,\xi,\tau)\exp(ik_z\xi)$, where $\xi=z$ and $\tau=t-k_z'z$ is the retarded time, the equation for the envelope E_1 can be obtained as a solvability condition in a multiple-scale asymptotic analysis of Eq. (6), which is detailed in the Appendix. Assuming that the QPM grating structure $\chi^{(2)}(z)$ is periodic with a period Λ satisfying the phase-matching condition $2\pi/\Lambda=|k_2-2k_z|$ (first-order QPM), one obtains the following envelope equation (see the Appendix):

$$2ik_z\frac{\partial E_1}{\partial \xi}=-\nabla_\perp^2E_1+\gamma\frac{\partial^2E_1}{\partial\tau^2}-i\beta\frac{\partial E_1}{\partial\tau}-\alpha E_1-\sigma_{eff}E_1^*, \tag{7}$$

where we have set

$$\alpha\equiv k_1^2-k_z^2, \tag{8}$$

$$\beta\equiv 2(k_1k_1'-k_zk_z'), \tag{9}$$

$$\gamma\equiv k_1k_1''+k_1'^2-k_z'^2, \tag{10}$$

$$\sigma_{eff}\equiv\left(\frac{\omega_0}{c_0}\right)^2E_2\langle\chi^{(2)}(z)\exp[i(k_2-2k_z)z]\rangle. \tag{11}$$

In Eq. (10), $k_1''=(\partial^2k/\partial\omega^2)_{\omega_0}$ is the second-order dispersion coefficient, whereas in Eq. (11) the brackets denote a spatial average. Note that, if we chose $k_z=k_1$ and $k_z'=k_1'$, one obtains $\alpha=\beta=0$, $\gamma=k_1k_1''$, and the envelope equation (7) reduces to its standard form.

III. PARAMETRIC AMPLIFICATION OF LOCALIZED ENVELOPE WAVES

In this section we study in detail the process of parametric amplification of localized envelope waves using either the extended paraxial wave equation (7) and the general equation (6), which accounts for nonparaxial effects and material dispersion at any order. As we will show in Sec. III A, the most interesting result of the analysis is that in the framework of the paraxial model [Eq. (7)] *distortionless* amplification of *phase-locked* envelope localized waves occurs. Weak wave distortions induced by higher-order dispersion terms and nonparaxial effects can be accounted for by means of a Fourier-Bessel analysis of Eq. (6), which is developed in Sec. III B. By considering the parametric amplification process in terms of polychromatic Bessel beams, distortionless amplification of localized waves is physically motivated by the coincidence between the dispersion relation of the cone angle of Bessel beams and the phase-matching condition of the parametric instability at any frequency, at least when higher-order dispersion effects can be neglected.

A. Localized envelope waves in the absence of gain

Before considering the process of parametric amplification, for the sake of clearness it is worth first considering the different types of envelope localized waves that can be supported in dispersive linear media [11,13,16,18]. By limiting our attention to waves with radial symmetry, the most general localized envelope wave that propagates undistorted with a phase velocity $v_f=\omega/k_z$ and a group velocity $v_g=1/k_z'$ can be found by a Fourier-Bessel analysis of Eq. (6) after setting $\sigma=0$ and can be written as $\mathcal{E}_1(r,z,t)=E_1(r,\tau)\exp(ik_zz)$, where $\tau=t-k_z'z$ is the retarded time, r is the radial coordinate, and $E_1(r,t)$ is the wave envelope, given by (see also [11,14,17])

$$E_1(r,\tau)=\int d\Omega S(\Omega)J_0[k_\perp(\Omega)r]\exp(-i\Omega\tau), \tag{12}$$

where $S(\Omega)$ is the spectral amplitude for Bessel beams and $k_\perp$ is the dispersion relation for the transverse wave number, which is given by

$$k_\perp(\Omega)=[k^2(\omega_0+\Omega)-(k_z+k_z'\Omega)^2]^{1/2}. \tag{13}$$

Equation (12) shows that the envelope wave is a superposition of monochromatic Bessel beams of spectral amplitude $S(\Omega)$ and with a frequency-dependent cone angle $\theta(\Omega)$ given by $\cos\theta(\Omega)=k_\perp(\Omega)/k(\omega_0+\Omega)$. To avoid divergences, the integral in Eq. (12) is extended over the frequency range such that $\theta(\Omega)$ is real valued. Note that if we consider material dispersion up to second order, i.e., we approximate the curve $k=k(\omega)$ by a parabola at around $\omega=\omega_0$, the dispersion relation (13) becomes

$$k_\perp(\Omega)=(\alpha+\beta\Omega+\gamma\Omega^2)^{1/2}, \tag{14}$$

where the coefficients α, β, and γ are given by Eqs. (8)–(10).

In the framework of the extended paraxial equation (7), an envelope localized wave $E_1(x,y,\tau)$ is a ξ-independent solution of Eq. (7) with $\sigma_{eff}=0$, i.e., it satisfies the equation

$$\mathcal{L}E_1\equiv\left(\nabla_\perp^2-\gamma\frac{\partial^2}{\partial\tau^2}+i\beta\frac{\partial}{\partial\tau}+\alpha\right)E_1=0. \tag{15}$$

A Fourier-Bessel analysis of Eq. (15) shows that its most general solution with axial symmetry is given again by Eq. (12), where the dispersion relation for the transverse wave number $k_\perp(\Omega)$ exactly matches the approximate relation given by Eq. (14). Different types of localized envelope waves are obtained, depending mainly on the sign of coefficients α, β and γ entering in Eq. (15), i.e., on the phase and group velocities of the wave. Here we are mainly interested in solutions with a constant envelope phase, e.g., real-valued envelope waves, which requires $S(\Omega)=S^*(-\Omega)$. In fact, as will be clearer below, the parametric down-conversion of pump field leads to simultaneous emission of phase-correlated photons at frequencies $\omega_0+\Omega$ and $\omega_0-\Omega$, and hence to avoid distortion effects the spectrum of the wave must be symmetric at around ω_0. There are mainly three distinct classes of real-valued envelope waves [13,16,18,22]. These are briefly reviewed here for the sake of clearness with reference to Fig. 1.

(i) *Pulsed Bessel beams* [13]. These localized waves are obtained when $\beta=\gamma=0$ and $\alpha>0$, i.e., for $k_z=k_1k_1'/(k_1'^2+k_1k_1'')^{1/2}$ and $k_z'=(k_1'^2+k_1k_1'')^{1/2}$. In this case, from Eq. (14) it follows that the transverse wave number $k_\perp$ is frequency independent, and Eq. (12) yields

$$E_1(r,\tau)=s(\tau)J_0\left(\sqrt{\frac{k_1^3k_1''}{k_1'^2+k_1k_1''}}r\right), \tag{16}$$

where $s(\tau)=\int d\Omega S(\Omega)\exp(-i\Omega\tau)$ is an arbitrary temporal profile. For instance, for a Gaussian spectral amplitude

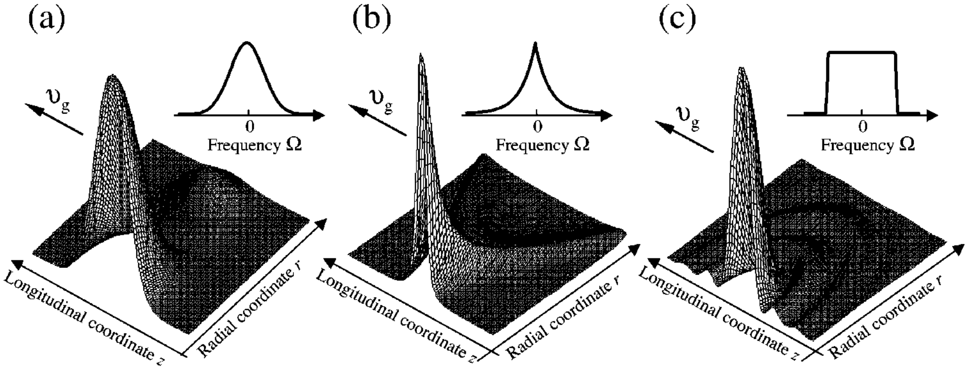

FIG. 1. Sketch of the intensity profile $|\mathcal{E}_1|^2$ and the corresponding spectrum $S(\Omega)$ (on the top) of the three different families of envelope localized waves discussed in the text: (a) pulsed Bessel beams; (b) X waves; and (c) sinc-shaped waves. The envelope propagates without spreading both in space and time with a group velocity $v_g = 1/k_z'$.

$S(\Omega) = [\tau_0/(2\pi^{1/2})]\exp(-\Omega^2\tau_0^2/4)$, one obtains a nondiffracting Bessel beam with a nondispersive Gaussian temporal profile (see Fig. 1, left picture):

$$E_1(r,\tau) = \exp(-\tau^2/\tau_0^2) J_0\left(\sqrt{\frac{k_1^3 k_1''}{k_1'^2 + k_1 k_1''}} r\right). \quad (17)$$

Note that the wave envelope satisfies a two-dimensional (2D) spatial Helmholtz equation, according to Eq. (15), and that pulsed Bessel beams exist solely in the normal dispersion region of the medium ($k_1'' > 0$).

(ii) *Envelope X waves* [16,22]. These localized waves are obtained when $\beta = 0$ and $\gamma > 0$. In this case, from Eq. (15) it follows that E_1 satisfies a 2D Klein-Gordon equation; X-shaped solutions for such an equation have been discussed, e.g., in [22]. A particularly interesting case [16] is that of luminal envelope X waves, corresponding to $\alpha = 0$, for which the localized wave is a solution of the scalar 2D wave equation. This occurs for $k_z = k_1$ and $k_z' = k_1'$. In this case from Eq. (12) one obtains the following representation of luminal X waves in terms of Bessel beams:

$$E_1(r,\tau) = \int d\Omega S(\Omega) J_0(\sqrt{k_1 k_1''}|\Omega| r)\exp(-i\Omega\tau). \quad (18)$$

In particular, for an exponential spectrum $S(\Omega) = (\tau_0/2)\exp(-\tau_0|\Omega|)$, one obtains [16] (see Fig. 1, middle picture):

$$E_1(r,\tau) = \mathrm{Re}\left\{\frac{\tau_0}{\sqrt{k_1 k_1'' r^2 + (\tau_0 + i\tau)^2}}\right\}. \quad (19)$$

Note that envelope luminal X waves exist in the normal dispersion region of the medium.

(iii) *Sinc-shaped envelope waves* [18]. These waves are obtained when $\beta = 0$, $\alpha > 0$, $\gamma < 0$ and, according to Eq. (15), they are solutions of a three-dimensional (3D) Helmholtz equation. These waves are always subluminal ($k_z' > k_1'$), and their spectral representation in terms of Bessel beams reads

$$E_1(r,\tau) = \int_{-\sqrt{\alpha/|\gamma|}}^{\sqrt{\alpha/|\gamma|}} d\Omega S(\Omega) J_0(\sqrt{\alpha - |\gamma|\Omega^2} r)\exp(-i\Omega\tau). \quad (20)$$

In particular, for a flat spectrum $S(\Omega)$ in the frequency interval $(-\sqrt{\alpha/|\gamma|}, \sqrt{\alpha/|\gamma|})$ one obtains a sinc-shaped envelope wave (see Fig. 1, right picture),

$$E_1(r,\tau) = \frac{\sin\sqrt{\alpha(r^2 + \tau^2/|\gamma|)}}{\sqrt{r^2 + \tau^2/|\gamma|}} \quad (21)$$

which is the simplest axially symmetric solution of the 3D Helmholtz equation in spherical coordinates.

B. Parametric amplification of the wave envelope: The extended paraxial model

Let us consider now the role of the parametric gain and assume that the extended paraxial envelope equation (7) may be applied. Let $A_{env}(r,\tau)$ be a *real-valued* localized envelope wave satisfying Eq. (15), and let us search for a solution to Eq. (7) in the form $E_1(r,\tau,\xi) = A_{env}(r,\tau)\mu(\xi)$; it then turns out that the amplitude μ satisfies the equation:

$$\frac{d\mu}{d\xi} = -\frac{\sigma_{eff}}{2ik_z}\mu^*. \quad (22)$$

If we assume the phase of the pump E_3, and hence of σ_{eff} [see Eq. (11)], such that the quantity $q_0 \equiv -\sigma_{eff}/(2ik_z)$ be real and positive, the solution to Eq. (22) with the initial condition $\mu(0) = \mu_0$ is given by

$$\mu(\xi) = \mathrm{Re}(\mu_0)\exp(q_0\xi) + i\,\mathrm{Im}(\mu_0)\exp(-q_0\xi). \quad (23)$$

We can hence conclude that an exponential amplification without distortion is realized under the phase locking condition $\mathrm{Im}(\mu_0) = 0$, the parameter $q_0 = |\sigma_{eff}|/(2k_z)$ being the parametric gain coefficient per unit length. The explicit expression of the gain coefficient q_0 in terms of more physical parameters reads

$$q_0 = \frac{d_{eff}\omega_0}{c_0 n_1}\sqrt{\frac{2I_2}{\epsilon_0 c_0 n_2}}, \quad (24)$$

where I_2 is the intensity of the pump wave, n_1 and n_2 are the refractive indices at the fundamental and second-harmonic frequencies, respectively, and $d_{eff} = (1/2)\langle\chi^{(2)}(z)\exp[i(k_2 - 2k_1)]\rangle$ is the effective nonlinear d coefficient of the second-order susceptibility. For instance, in case of a $\pm$ squareshaped QPM grating structure, one has $d_{eff}=(2/\pi)d$, where $d=(1/2)\chi^{(2)}$ is the relevant element of the d-tensor involved in the interaction process.

C. Parametric amplification of polychromatic Bessel beams: The general case

We turn back to the general Eq. (6) and study the process of parametric amplification for a localized envelope wave by taking into account nonparaxial effects and material dispersion at any order. To this aim, let us search for a solution of Eq. (6) as a superposition of Bessel beams in the form

$$\mathcal{E}_1(r,\tau,z)=\left\{\int d\Omega[H(\Omega,z)\exp(-i\Omega\tau)+G^*(\Omega,z)\exp(i\Omega\tau)]J_0[k_\perp(\Omega)r]\right\}\exp(ik_z z), \tag{25}$$

where $\tau=t-k_z' z$, $k_\perp$ is defined by Eq. (13), and the initial condition $H(\Omega,0)=S(\Omega)$ and $G(\Omega,0)=0$ are assumed, where $S(\Omega)$ is the spectral amplitude of the incident localized wave [see Eq. (12)]. In the absence of parametric gain, H and G are independent of z, i.e., $H=S$ and $G=0$, and the dispersionless and diffractionless solution given by Eq. (12) is retrieved. In the presence of the parametric gain, the evolution equations for H and G can be easily derived after the substitution of Eq. (25) into Eq. (6) and setting equal the terms oscillating at the same frequency. This yields

$$\frac{\partial^2 H}{\partial z^2}+2i(k_z+\Omega k_z')\frac{\partial H}{\partial z}=-\sigma(z)G\exp[i(k_2-2k_z)z], \tag{26}$$

$$\frac{\partial^2 G}{\partial z^2}+2i(-k_z+\Omega k_z')\frac{\partial G}{\partial z}=-\sigma^*(z)H\exp[-i(k_2-2k_z)z]-\zeta(\Omega)G, \tag{27}$$

where the detuning term $\zeta(\Omega)$ is given by $\zeta(\Omega)\equiv-k_\perp^2(\Omega)+k_\perp^2(-\Omega)$. Note that if we consider material dispersion up to second order, for which $k_\perp(\Omega)$ is approximated by Eq. (14), the detuning term ζ vanishes for the three types of localized waves considered in Sec. III A. If higher-order dispersion terms are accounted for, the detuning term does not vanish; however, one may assume $|\zeta|/k_z^2\ll 1$. In this case, and for a small gain term ($|\sigma|/k_z^2\ll 1$), one can solve Eqs. (26) and (27) perturbatively by a standard multiple scale (or averaging) method. Assuming again that the QPM grating structure $\chi^{(2)}$ is periodic with a period Λ satisfying the phase-matching condition $2\pi/\Lambda=|k_2-2k_z|$ (first-order QPM), the evolution equations for the amplitudes H and G at leading order in the perturbative analysis read

$$2i(k_z+\Omega k_z')\frac{\partial H}{\partial z}=-\sigma_{eff}G, \tag{28}$$

$$2i(-k_z+\Omega k_z')\frac{\partial G}{\partial z}=-\sigma_{eff}^* H-\zeta G, \tag{29}$$

where σ_{eff} is given by Eq. (11). The solution to Eqs. (28) and (29) with the initial condition $H(\Omega,0)=S(\Omega)$, $G(\Omega,0)=0$ is given by

$$H(\Omega,z)=\frac{S(\Omega)}{\lambda_+-\lambda_-}[\lambda_+\exp(\lambda_- z)-\lambda_-\exp(\lambda_+ z)], \tag{30}$$

$$G(\Omega,z)=-\frac{\lambda_+\lambda_-(k_z+\Omega k_z')S(\Omega)}{k_z q_0(\lambda_+-\lambda_-)}\times[\exp(\lambda_+ z)-\exp(\lambda_- z)], \tag{31}$$

where

$$\lambda_\pm=\frac{-i\zeta(k_z+\Omega k_z')\pm\sqrt{-\zeta^2(k_z+\Omega k_z')^2+16|q_0|^2k_z^2(k_z^2-\Omega^2 k_z'^2)}}{4(k_z^2-\Omega^2k_z'^2)} \tag{32}$$

and $q_0=-\sigma_{eff}/(2ik_z)$, which is assumed to be real valued and positive without loss of generality [see Eq. (24)]. Equations (25), (30), and (31) provide the most general solution to the problem of parametric amplification of localized envelope waves, and account for distortion effects due to higher-order material dispersion terms, asymmetric wave spectrum, and nonparaxiality. In particular, if we assume that the detuning term ζ can be neglected $[k_\perp(\Omega)\simeq k_\perp(-\Omega)]$ and the wave spectrum $S(\Omega)$ satisfies the symmetry condition $S(\Omega)=S^*(-\Omega)$, which occurs for the three types of localized waves considered in Sec. III A, one obtains

$$\mathcal{E}_1(r,\tau,z)=\left\{\int d\Omega g(\Omega,z)S(\Omega)\times\exp(-i\Omega\tau)J_0[k_\perp(\Omega)r]\right\}\exp(ik_z z), \tag{33}$$

where we have introduced the spectral gain function:

$$g(\Omega,z)=\cosh\left(\frac{q_0 z}{\sqrt{1-(\Omega k_z'/k_z)^2}}\right)+\sqrt{\frac{1-\Omega k_z'/k_z}{1+\Omega k_z'/k_z}}\sinh\left(\frac{q_0 z}{\sqrt{1-(\Omega k_z'/k_z)^2}}\right). \tag{34}$$

Note that, in the paraxial limit, which implies $|\Omega k_z'/k_z| \ll 1$ within the spectral extent of the wave, from Eq. (34) one obtains $g \simeq \exp(q_0 z)$, i.e., one has exponential amplification without wave distortion according to the analysis of Sec. III B.

In order to provide some numerical insights to theoretical analysis, let us consider parametric amplification of envelope localized waves in periodically poled lithium niobate (PPLN). At a signal wavelength (in vacuum) $\lambda_1 = 1550$ nm of optical communications, corresponding to a pump wave at $\lambda_2 = 775$ nm, the material shows normal group-velocity dispersion ($k_1 = 8.6709 \times 10^6$, $k_1' = 7.281 \times 10^{-9}$, $k_1'' = 9.836 \times 10^{-26}$, SI units; the dispersion properties of lithium niobate have been calculated by means of a Sellmeier equation according to [25]). For extraordinary waves, one has $2\pi/|k_2 - 2k_1| \simeq 19$ μm and $d = d_{33} \simeq 27$ pm/V. The type of envelope localized wave that can be amplified depends on the QPM grating period Λ, which fixes the longitudinal wave number k_z, i.e., the phase velocity of the carrier, according to the phase matching condition $k_2 - 2k_z = \pm 2\pi/\Lambda$, and hence the group velocity $v_g = 1/k_z' = k_z/(k_1 k_1')$ from the requirement $\beta = 0$ [see Eq. (9)]. For the branch $k_2 - 2k_z = -2\pi/\Lambda$, it turns out that the coefficients α and γ, given by Eqs. (8) and (11), are always positive, i.e., phase matching is realized solely for envelope X waves. Conversely, for the branch $k_2 - 2k_z = 2\pi/\Lambda$, γ turns out to be positive for $\Lambda > 13.634$ μm and negative for $\Lambda < 13.634$ μm, whereas α is positive for $\Lambda > 19.454$ μm and negative for $\Lambda < 19.454$ μm. Thus in this case phase matching is realized for sinc-shaped waves when $\Lambda < 13.634$ μm and for X waves when $\Lambda > 13.634$ μm. In particular, luminal envelope X waves are phase matched at $\Lambda = 19.454$ μm, for which α vanishes. At $\Lambda = 13.634$ μm, γ vanishes and phase matching is realized for pulsed Bessel beams. As an example, Figs. 2 and 3 show parametric amplification in PPLN of luminal envelope X waves corresponding to an exponential spectrum with two different values of τ_0 [see Eq. (19)]. Figures 2(a)and 3(a), and 2(b) and 3(b) show a gray-scale representation of the amplitude $|\mathcal{E}_1(r,t)|$ of the envelope wave at the entrance (a) and at the exit (b) of the crystal, the propagation being calculated taking into account dispersion effects at any order by means of Eqs. (25), (30), and (31) and by numerical computation of the integrals entering in Eq. (25). Figures 2(c) and 3(c) show the temporal behavior of the envelope intensity on the axis, i.e., for $r = 0$, at the exit of the crystal, and compared with that predicted in the absence of distortion and using the approximate expression for the spectral gain given by Eqs. (33) and (34), which neglect the detuning term $\zeta(\Omega)$.

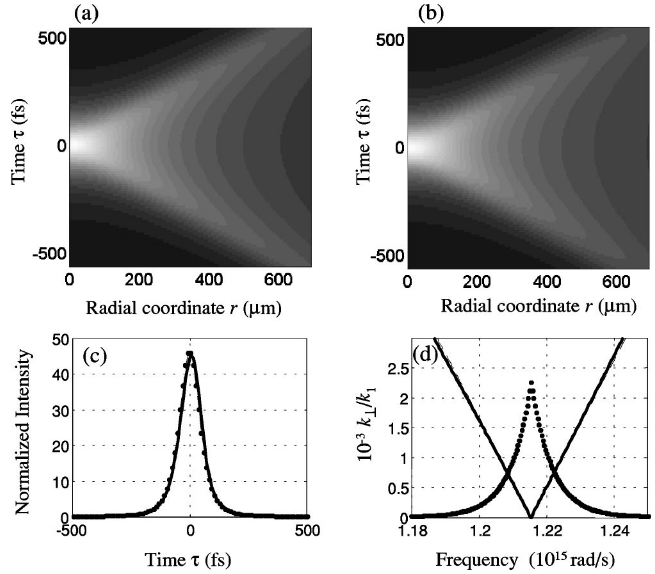

FIG. 2. Parametric amplification of luminal X waves in PPLN. QPM grating period $\Lambda \simeq 19.5$ μm, pump intensity $I_2 = 400$ kW/cm^2, crystal length $z = 5$ cm [corresponding to $q_0 = 38.3695$ m^{-1} and to a parametric gain $\exp(q_0 z) = 6.8106$], and pulse duration parameter $\tau_0 = 80$ fs. (a) Snapshot of the amplitude of luminal X wave in the (r,τ) plane at the entrance of the crystal. (b) Snapshot of the amplitude of the X wave at the exit of the crystal calculated by taking into account dispersion effects at any order. (c) Behavior of the on-axis field intensity $|\mathcal{E}(r=0,\tau)|^2$ versus retarded time τ at the exit of the crystal (solid line); the dashed line in the figure, almost overlapped with the solid one, is the corresponding behavior predicted by Eqs. (33) and (34), i.e., by assuming $k_\perp(\Omega) \simeq k_\perp(-\Omega)$, whereas the dotted line is the behavior predicted neglecting distortion effects. The intensity is normalized to the peak intensity of the X wave at the entrance plane of the crystal. (d) Behavior of the dispersion relation $k_\perp(\omega)$ of the X wave [Eq. (13)], normalized to k_1: exact curve (solid line) and approximate curve (dashed line) given by Eq. (14). The dotted line shows the exponential spectral amplitude $S(\omega)$ of the X wave.

Note that, as for relatively long pulses, such as in Fig. 2 ($\tau_0 = 50$ fs), distortionless amplification is obtained according to the theoretical analysis, for short pulses, as in the case of Fig. 3 ($\tau_0 = 15$ fs) distortion effects are clearly visible. The main contribution to pulse distortions can be ascribed to higher-order dispersion effects, remarkably to the asymmetry of the dispersion relation $k_\perp(\Omega)$ around $\Omega = 0$ [see Figs. 2(d) and 3(d)], leading to a nonvanishing detuning term $\zeta(\Omega)$.

D. Parametric instability and localized envelope waves

The previous analysis has been concerned with the parametric amplification problem, in which a signal wave is seeded into the nonlinear medium; however, a related issue is that of parametric fluorescence, i.e., the growth from noise of the signal wave at frequency ω_0 due to the parametric instability. It is not the aim of this paper to study in detail this problem; however, it is worth pointing out here that the axially symmetric perturbations that maximize the growth rate of the parametric instability reproduce the characteristic dispersion relation of envelope localized waves given by Eq.

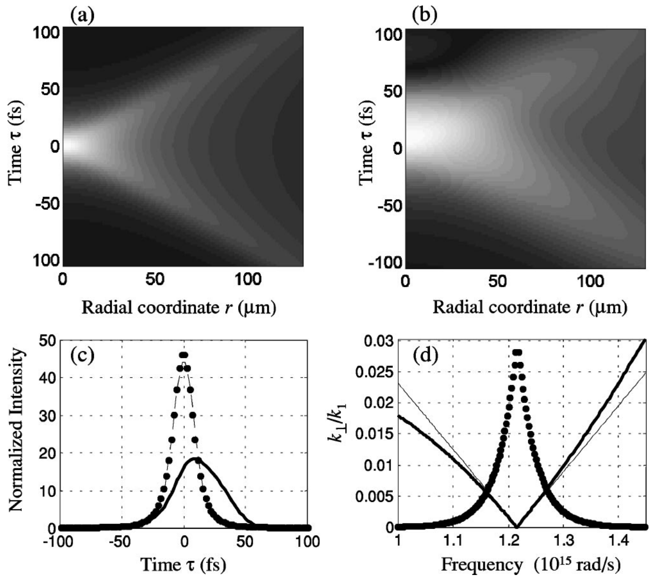

FIG. 3. Same as Fig. 2, but for a pulse duration parameter $\tau_0 = 15$ fs.

(14) with $\beta = 0$. This means that, depending on the periodicity of the QPM grating, envelope X waves or sinc-shaped waves might be spontaneously generated by the parametric instability. A similar result has been recently predicted in case of modulational instability of plane waves in a nonlinear Kerr medium [22]. To determine the perturbations with maximum growth rate, let us consider axially symmetric perturbations which have a Fourier-Bessel representation according to

$$\mathcal{E}_1(r,z,t) = \int_{-\infty}^{\infty} d\Omega \int_0^{\infty} d\kappa \kappa S(\Omega,\kappa,z) J_0(\kappa r) \exp(-i\Omega t). \tag{35}$$

After setting Eq. (35) into Eq. (6), one obtains the following equation for the spectral amplitude $S(\Omega,\kappa,z)$:

$$\frac{\partial^2 S(\Omega)}{\partial z^2} + [k^2(\omega_0+\Omega) - \kappa^2] S(\Omega) = -\sigma(z)\exp(ik_2 z) S^*(-\Omega). \tag{36}$$

Since the coefficients in this equation are periodic in z, one can apply Floquet theory and determine the Floquet exponents, i.e., the growth rate of perturbations, which depend on Ω and κ. Assuming again the paraxial limit $\kappa/k_1 \ll 1$, the low-gain limit $|\sigma|/k_1^2 \ll 1$ and expanding $k(\omega_0+\Omega)$ up to second order in Ω, one can derive approximate expressions of Floquet exponents by standard multiple scale or averaging techniques. After setting $S(\Omega,\kappa,z) = A(\Omega,\kappa,z)\exp[i(k_z + k_z'\Omega)z]$, where the amplitude A is assumed to vary slowly with respect to z as compared to the exponential term and $k_z' = k_1' k_z / k_1$, one finds that A satisfies the amplitude equation:

$$2ik_z \frac{\partial A(\Omega,\kappa,z)}{\partial z} = \rho(\Omega) A(\Omega,\kappa,z) - \sigma_{eff} A^*(-\Omega,\kappa,z), \tag{37}$$

where σ_{eff} is given by Eq. (11), $\rho(\Omega,\kappa) \equiv k_z^2 - k_1^2 + \kappa^2 - (k_1 k_1'' + k_1'^2 - k_z'^2)\Omega^2$, and for first-order QPM k_z is determined by the phase matching condition $k_2 - 2k_z = \pm 2\pi/\Lambda$. The Lyapunov exponents associated with Eq. (37) can be easily calculated and reads

$$\lambda_\pm = -i\frac{\rho(\Omega,\kappa) - \rho(-\Omega,\kappa)}{4k_z} \pm \frac{1}{4k_z}\sqrt{4|\sigma_{eff}|^2 - [\rho(\Omega,\kappa) + \rho(-\Omega,\kappa)]^2}. \tag{38}$$

The growth rate is thus maximized, reaching the value $q_0 = |\sigma_{eff}|/2k_z$, when the phase-matching condition $\rho(\Omega,\kappa) + \rho(-\Omega,\kappa) = 0$ is satisfied, which in terms of parameters κ and Ω reads explicitly

$$\kappa = [k_1^2 - k_z^2 + (k_1 k_1'' + k_1'^2 - k_z'^2)\Omega^2]^{1/2}. \tag{39}$$

Note that Eq. (39) exactly reproduces the dispersion relation $k_\perp = k_\perp(\Omega)$ for localized envelope waves given by Eq. (14) after noting that $\beta = 0$. This proves that the dispersion relation for the cone angle of polychromatic Bessel beams forming the envelope localized waves discussed in Sec. III A exactly matches the phase matching condition for maximum parametric gain provided that the condition $k_z' = k_1' k_z / k_1$, relating envelope and phase velocities of the wave, is satisfied.

IV. CONCLUSIONS AND DISCUSSION

In this paper we have studied the process of optical parametric amplification of localized envelope waves at carrier frequency ω_0 in dispersive and nonlinear $\chi^{(2)}$ media pumped by a plane-wave and continuous-wave pump field at frequency $2\omega_0$ in the undepleted pump approximation. The phase matching condition for efficient parametric down-conversion of pump photons, which can be controlled by a periodic QPM grating structure, determines the wave number k_z of the localized envelope wave for maximum parametric gain, i.e., its phase velocity $v_f = \omega_0/k_z$, whereas the group velocity $v_g = 1/k_z'$ of the envelope is determined by the additional constraint $k_1 k_1' = k_z k_z'$ [$k_1 = k(\omega_0)$ and $k_1' = (\partial k/\omega)_{\omega_0}$ are fixed by the material dispersion properties], which arises from the simultaneous emission of photons at frequencies $\omega_0 \pm \Omega$ in the down conversion of pump photons at frequency $2\omega_0$. The parametric amplification of different kinds of localized envelope waves that satisfy such a constraint, including pulsed Bessel beams, envelope X waves, and sinc-shaped envelope waves, has been studied in detail both by the derivation of a generalized envelope equation in the paraxial and near-monochromatic limits and by a direct analysis of the general nonparaxial equation using a polychromatic Bessel beam expansion. The issue of parametric instability and envelope localized waves has been addressed as well. It is envisaged that, similarly to what was found in the case of modulational instability of plane waves in Kerr media [22], envelope localized waves might be spontaneously generated from noise due to the parametric instability.

APPENDIX: DERIVATION OF THE GENERALIZED PARAXIAL ENVELOPE EQUATION

In this appendix we derive the envelope equation (7) by a multiple scale asymptotic analysis of Eq. (6) in the paraxial approximation. First of all, it is worth rewriting Eq. (6) in terms of dimensionless parameters and spatial-temporal variables to highlight the order of magnitude of various terms entering in the equation. After introducing the dimensionless variables $(x',y',z')=k_z\times(x,y,z)$ and $t'=\omega_0 t$, where k_z is a wave number close to (but not necessarily coincident with) $k_1=k(\omega_0)$, Eq. (6) can be cast in the form

$$\frac{\partial^2\mathcal{E}_1}{\partial z'^2}+\nabla_\perp^2\mathcal{E}_1+\frac{k^2\left[\omega_0\left(1+i\frac{\partial}{\partial t'}\right)\right]}{k_z^2}\mathcal{E}_1 =-\frac{\sigma(z')}{k_z^2}\mathcal{E}_1^*\exp(ik_2z'/k_z), \tag{A1}$$

where the transverse Laplacian is now applied to the (x',y') spatial variables. The paraxial and quasimonochromatic approximations imply that $\mathcal{E}_1$ varies slowly with respect to x', y' and t', so that we assume that $\mathcal{E}_1$ depends on x', y' and t' through the slow variables $X=\epsilon x'$, $Y=\epsilon y'$, and $T=\epsilon t'$, where ϵ is a smallness parameter that is assumed as a bookkeeping parameter that organizes the asymptotic analysis. The slow dependence of $\mathcal{E}_1$ on t' allows one to make a power expansion of the operator $k^2[\omega_0(1+i\partial_{t'})]$ entering in Eq. (A1); up to order $\sim\epsilon^2$, one can set

$$\frac{k^2\left[\omega_0\left(1+i\frac{\partial}{\partial t'}\right)\right]}{k_z^2}=\frac{k_1^2}{k_z^2}+2i\omega_0\frac{k_1k_1'}{k_z^2}\frac{\partial}{\partial t'} -\omega_0^2\frac{k_1k_1''+k_1'^2}{k_z^2}\frac{\partial^2}{\partial t'^2}, \tag{A2}$$

where $k_1'=(\partial k/\partial\omega)_{\omega_0}$ and $k_1''=(\partial^2k/\partial\omega^2)_{\omega_0}$. In addition, we consider the small parametric gain limit by assuming in Eq. (A1) $\sigma/k_z^2\sim O(\epsilon^2)$. Such a choice of scaling for σ/k_z^2 is motivated by the need to make diffraction, dispersion, and parametric gain terms appearing in Eq. (A1) of the same order of magnitude. Our aim is now to search for a solution of Eq. (A1) as a power expansion in ϵ,

$$\mathcal{E}_1=\mathcal{E}_1^{(0)}+\epsilon\mathcal{E}_1^{(1)}+\epsilon^2\mathcal{E}_1^{(2)}+\cdots, \tag{A3}$$

such that at leading order $\mathcal{E}_1^{(0)}$ describes a wave propagating with a phase velocity $v_f=\omega_0/k_z$ and an envelope velocity $v_g=1/k_z'$, where k_z and k_z' are free parameters close to, but not necessarily coincident with, $k_1=k(\omega_0)$ and $k_1'=(\partial k/\partial\omega)_{\omega_0}$, respectively. For consistency, it turns out that one has to assume $(k_z-k_1)/k_1\sim O(\epsilon^2)$ and $(k_z'-k_1')/k_1'\sim O(\epsilon)$. Finally, slow spatial variables $Z_0=z'$, $Z_1=\epsilon z'$, $Z_2=\epsilon^2z'$ are introduced to avoid the occurrence of secular growing terms in the asymptotic expansion. With such considerations in mind and using Eq. (A2), Eq. (A1) can be written in the form

$$\frac{\partial^2\mathcal{E}_1}{\partial z'^2}+\mathcal{E}_1=-2i\omega_0\frac{k_z'}{k_z}\frac{\partial\mathcal{E}_1}{\partial t'} +\left[1-\frac{k_1^2}{k_z^2}-2i\omega_0\left(\frac{k_1k_1'}{k_z^2}-\frac{k_z'}{k_z}\right)\frac{\partial}{\partial t'} +\frac{\omega_0^2(k_1k_1''+k_1'^2)}{k_z^2}\frac{\partial^2}{\partial t'^2}-\nabla_\perp^2\right]\mathcal{E}_1 -\frac{\sigma}{k_z^2}\mathcal{E}_1^*\exp(ik_2z'/k_z). \tag{A4}$$

Note that with the chosen scaling, the last term in Eq. (A4) and the operator in the square bracket on the right hand side turn out to be of order $\sim\epsilon^2$, the first term on the right hand side in Eq. (A4) is of order $\sim\epsilon$, and the terms on the left hand side are of order $\sim\epsilon^0$. Substitution of the power expansion (A3) into Eq. (A4), using the derivative rule $\partial_{z'}^2=\partial_{Z_0}^2+2\epsilon\partial_{Z_0}\partial_{Z_1}+\epsilon^2(\partial_{Z_1}^2+2\partial_{Z_0}\partial_{Z_2})$ and setting equal the terms of the same order in ϵ, a hierarchy of equations for successive corrections to $\mathcal{E}_1$ is obtained. At leading order one has

$$\frac{\partial^2\mathcal{E}_1^{(0)}}{\partial Z_0^2}+\mathcal{E}_1^{(0)}=0, \tag{A5}$$

the solution of which, for progressive waves, is

$$\mathcal{E}_1^{(0)}=E_1(X,Y,T,Z_1,Z_2)\exp(iZ_0), \tag{A6}$$

where the amplitude E_1 is an arbitrary function of slow variables X,Y,T,Z_1, and Z_2. The evolution equations of the amplitude E_1 on the spatial scales Z_1 and Z_2 are obtained as solvability conditions in the asymptotic analysis at orders $\sim\epsilon$ and $\sim\epsilon^2$, respectively. At $O(\epsilon)$ one obtains

$$\frac{\partial^2\mathcal{E}_1^{(1)}}{\partial Z_0^2}+\mathcal{E}_1^{(1)}=U^{(1)}, \tag{A7}$$

where

$$U^{(1)}\equiv-2\frac{\partial^2\mathcal{E}_1^{(0)}}{\partial Z_0\partial Z_1}-2i\omega_0\frac{k_z'}{k_z}\frac{\partial\mathcal{E}_1^{(0)}}{\partial T} =-2i\left(\frac{\partial E_1}{\partial Z_1}+\omega_0\frac{k_z'}{k_z}\frac{\partial E_1}{\partial T}\right)\exp(iZ_0). \tag{A8}$$

To avoid the occurrence of secular growths, terms oscillating like $\sim\exp(\pm iZ_0)$ appearing in $U^{(1)}$ should vanish. This yields the following solvability condition:

$$\frac{\partial E_1}{\partial Z_1}+\omega_0\frac{k_z'}{k_z}\frac{\partial E_1}{\partial T}=0, \quad \text{(A9)}$$

and one can assume $\mathcal{E}_1^{(1)}=0$ as a solution at this order. The physical meaning of Eq. (A9) is that, at leading order, the envelope E_1 propagates undistorted at a group velocity $v_g=1/k_z'$ in terms of physical variables.

At $O(\epsilon^2)$ one obtains

$$\frac{\partial^2\mathcal{E}_1^{(2)}}{\partial Z_0^2}+\mathcal{E}_1^{(2)}=U^{(2)}, \quad \text{(A10)}$$

where

$$U^{(2)}\equiv-2\frac{\partial^2\mathcal{E}_1^{(0)}}{\partial Z_0\partial Z_2}-\frac{\partial^2\mathcal{E}_1^{(0)}}{\partial Z_1^2}+\left(1-\frac{k_1^2}{k_z^2}-\nabla_\perp^2\right)\mathcal{E}_1^{(0)} -\frac{2i\omega_0}{k_z}\left(\frac{k_1k_1'}{k_z}-k_z'\right)\frac{\partial\mathcal{E}_1^{(0)}}{\partial T}+\left(\frac{\omega_0}{k_z}\right)^2(k_1k_1''+k_1'^2) \times\frac{\partial^2\mathcal{E}_1^{(0)}}{\partial T^2}-\frac{\sigma(Z_0)}{k_z^2}\mathcal{E}_1^{(0)*}\exp(ik_2Z_0/k_z). \quad \text{(A11)}$$

The solvability condition at this order is obtained by imposing that the term oscillating like $\exp(iZ_0)$ in the expression of $U^{(2)}$ vanish. Note that the parametric gain contribution to the forcing term $U^{(2)}$ depends on Z_0 by the term $\exp[i(k_2/k_z-1)Z_0]\sigma(Z_0)$, and hence it yields a nonvanishing contribution to the solvability condition provided that the spatial period Λ of $\sigma(z)$ satisfies the phase-matching condition $|k_2-2k_z|\simeq 2m\pi/\Lambda$ (m is an integer number) and $\sigma(z)$ has a nonvanishing Fourier component of order m. Slight deviations from perfect phase matching can be accounted for by allowing $\sigma(z')$ to depend slowly on the spatial scale Z_2. In the following we will consider first-order QPM, i.e., $m=1$, and assume perfect phase matching. In this case the solvability condition at $O(\epsilon^2)$ reads

$$-2i\frac{\partial E_1}{\partial Z_2}-\frac{\partial^2 E_1}{\partial Z_1^2}+\left(1-\frac{k_1^2}{k_z^2}-\nabla_\perp^2\right)E_1-\frac{2i\omega_0}{k_z}\left(\frac{k_1k_1'}{k_z}-k_z'\right) \times\frac{\partial E_1}{\partial T}+\left(\frac{\omega_0}{k_z}\right)^2(k_1k_1''+k_1'^2)\frac{\partial^2E_1}{\partial T^2}-\frac{\sigma_{eff}}{k_z^2}E_1^*=0, \quad \text{(A12)}$$

where σ_{eff} is the relevant Fourier coefficient of $\sigma(z)$, given by Eq. (11) in the text. Taking into account that $\partial^2E_1/\partial Z_1^2=(\omega_0k_z'/k_z)^2\partial^2E_1/\partial T^2$ [see Eq. (A9)], Eq. (A12) yields

$$\frac{\partial E_1}{\partial Z_2}=\frac{1}{2i}\left(1-\frac{k_1^2}{k_z^2}-\nabla_\perp^2\right)E_1-\frac{\omega_0}{k_z}\left(\frac{k_1k_1'}{k_z}-k_z'\right)\frac{\partial E_1}{\partial T} +\frac{1}{2i}\left(\frac{\omega_0}{k_z}\right)^2(k_1k_1''+k_1'^2-k_z'^2)\frac{\partial^2E_1}{\partial T^2}-\frac{\sigma_{eff}}{2ik_z^2}E_1^*. \quad \text{(A13)}$$

If we stop the asymptotic expansion at this order, the evolution equation of the amplitude E_1 reads $\partial E_1/\partial z'=\epsilon\partial E_1/\partial Z_1+\epsilon^2\partial E_1/\partial Z_2$. Using Eqs. (A9) and (A13), after reintroducing the original physical variables $(x,y,z)=(1/k_z)\times(x',y',z')$, $t=t'/\omega_0$ and setting $\epsilon=1$, one finally obtains the following amplitude equation for the envelope $E_1(x,y,z,t)$:

$$2ik_z\left(\frac{\partial E_1}{\partial z}+k_z'\frac{\partial E_1}{\partial t}\right)=-\nabla_\perp^2E_1+\gamma\frac{\partial^2E_1}{\partial t^2} -i\beta\frac{\partial E_1}{\partial t}-\alpha E_1-\sigma_{eff}E_1^*, \quad \text{(A14)}$$

where the coefficients α, β, and γ are given by Eqs. (8)–(10) in the text. In the reference frame traveling at the envelope velocity $v_g=1/k_z'$, i.e. in the transformed variables $\xi=z$ and $\tau=t-k_z'z$, one obtains the envelope equation (7) given in the text.

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